1 Title

Mono-anabelian geometry over sub-p-adic fields via Belyi cuspidalization

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- Professor Mochizuki, [AbsTpIII], §1

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2 Abstract

In this talk, we study mono-anabelian geometry. In more concrete terms, we prove the following assertion ([AbsTpIII], Theorem 1.9): Let k_0 be a number field, $k \supseteq k_0$ a sub-p-adic field, $\bar{k}$ an algebraic closure of k, and U_0/k_0 a hyperbolic curve which is isogenous to a hyperbolic curve of genus zero. Write $\bar{k}_0$ for the algebraic closure of k_0 in $\bar{k}$. Then we reconstruct group-theoretically the function field $\mathrm{Fnct}(U_0 \times_{k_0} \bar{k}_0)$ from

$$1 \to \pi_1(U_0 \times_{k_0} \bar{k}) \to \pi_1(U_0 \times_{k_0} k) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

(regarded as an exact sequence of abstract profinite groups) via the technique of Belyi cuspidalization.

$$\pi_1(U_0 \times_{k_0} k) \twoheadrightarrow \mathrm{Gal}(\bar{k}/k) \rightsquigarrow \mathrm{Fnct}(U_0 \times_{k_0} \bar{k}_0)$$

3 Notation

p: a prime number

k_0: an NF (finite extension of $\mathbb{Q}$)

$k \supseteq k_0$: sub-p-adic ($k \overset{\exists}{\hookrightarrow} \exists$ finitely generated$/\mathbb{Q}_p$)

$\bar{k}$: an algebraic closure of k

$\bar{k}_0$: the algebraic closure of k_0 in $\bar{k}$

$X_0^{\log}/k_0$: hyperbolic log curve

X_0: the underlying scheme of $X_0^{\log}$

U_{X_0}: the interior of $X_0^{\log}$

$X^{\log} \overset{\text{def}}{=} X_0^{\log} \times_{k_0} k$

$G_k \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}/k)$

$G_{k_0} \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}_0/k_0)$

Thus, X_0 is a proper curve, and $U_{X_0} \subseteq X_0$ is a hyperbolic curve

Suppose that U_{X_0}/k_0 is isogenous to a hyperbolic curve of genus zero.

Today, we consider X_0, U_{X_0}

4 Main theorem

Today, we consider semi-absolute mono profinite Grothendieck conjecture

Main Theorem ([AbsTpIII], (1.9))

We reconstruct group-theoretically the function field $\mathrm{Fnct}(U_{X_0} \times_{k_0} \bar{k}_0)$ from

$$1 \to \pi_1(U_{X_0} \times_{k_0} \bar{k}) \to \pi_1(U_X) \to G_k \to 1$$

$$\pi_1(U_X) \twoheadrightarrow G_k \rightsquigarrow \mathrm{Fnct}(U_{X_0} \times_{k_0} \bar{k}_0)$$

Remark ([AbsTpIII], (1.9.2))

If k: an MLF (finite extension of $\mathbb{Q}_{\exists p}$) or an NF, then

$$\pi_1(U_X) \rightsquigarrow \mathrm{Fnct}(U_{X_0} \times_{k_0} \bar{k}_0)$$

Remark

If k: an NF (so $k = k_0$), then

$$\pi_1(U_X) \rightsquigarrow G_{k_0} \curvearrowright \mathrm{Fnct}(U_{X_0} \times_{k_0} \bar{k}_0)$$

5 Notation2

We may assume without loss of generality that

- X_0/k_0 of genus ≥ 2
- k_0 is algebraic closed in k

Let $S_0 \subseteq X_0^{\mathrm{cl}}$: a finite subset (cl = "the set of closed points")
$X_{\mathrm{NF}}^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \mathrm{Im}(X_0^{\mathrm{cl}} \hookrightarrow X^{\mathrm{cl}})$
$X(\bar{k}) \subseteq X^{\mathrm{cl}}$: the set of $\bar{k}$-value points
$S \stackrel{\mathrm{def}}{=} \mathrm{Im}(S_0 \subseteq X_0^{\mathrm{cl}} \stackrel{\sim}{\to} X_{\mathrm{NF}}^{\mathrm{cl}})$

$$\begin{array}{ccc} S_0 & \xrightarrow{\sim} & S \\ \cap & & \cap \\ X_0^{\mathrm{cl}} & \xrightarrow{\sim} & X_{\mathrm{NF}}^{\mathrm{cl}} \\ & & \cap \\ & & X^{\mathrm{cl}} \\ & & \cup \\ & & X(\bar{k}) \end{array}$$

6 Reconstruct Galois groups

[AbsTpI], (2.6), (v), (vi),

START: $\pi_1(U_X)$

If k is an NF or an MLF, then

$$\pi_1(U_X) \rightsquigarrow \pi_1(U_X) \twoheadrightarrow G_k$$

7 Belyi cuspidalization-1

[AbsTpII], (3.6)(3.7)(3.8)

START: $\pi_1(U_X) \twoheadrightarrow G_k$

We want to reconstruct

$$\{\pi_1(U_X \setminus T) \to \pi_1(U_X)\}_{T \subseteq X^{\mathrm{cl}}_{\mathrm{NF}}:\ \text{finite subset}} \quad \text{and} \quad X^{\mathrm{cl}}_{\mathrm{NF}}$$

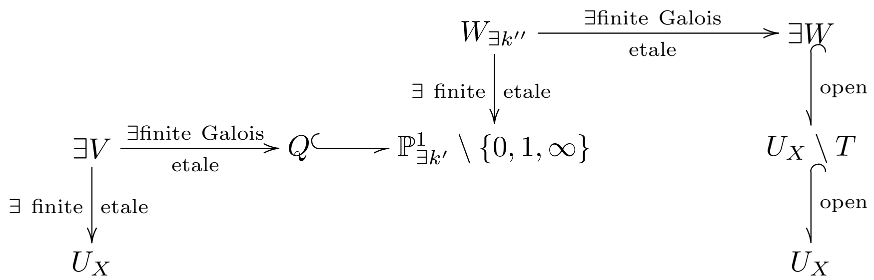
Since U_{X_0} is isogenous to genus 0

$$\begin{array}{ccccc} \exists V & \xrightarrow[\text{etale}]{\exists \text{finite Galois}} & Q & \hookrightarrow & \mathbb{P}^1_{\exists k'} \setminus \{0, 1, \infty\} \\ {\scriptstyle \exists\ \text{finite}} \downarrow {\scriptstyle \text{etale}} & & & & \\ U_X & & & & \end{array}$$

where k'/k; finite extension

8 Belyi cuspidalization-2

By the existence of Belyi maps



where k''/k': finite extension, k''/k: Galois.

By using Belyi cuspidalization,

$$\rightsquigarrow \{\pi_1(U_X \setminus T) \to \pi_1(U_X)\}_{T \subseteq X^{\mathrm{cl}}_{\mathrm{NF}}:\ \text{finite subset}}$$

9 Belyi cuspidalization-3

By Nakamura (cf. [AbsTpI], (4.5)),

$$\pi_1(U_X \setminus T) \rightsquigarrow \{I_P \subseteq \pi_1(U_X \setminus T)\}_{P \in (X \setminus (U_X \setminus T))(\bar{k})}$$

where I_P denotes an inertia group

we consider $\pi_1(U_X \setminus T)$-conjugacy class of inertia groups

$$\rightsquigarrow X^{\mathrm{cl}}_{\mathrm{NF}}$$

10 Decomposition groups

START: $\pi_1(U_X) \twoheadrightarrow G_k$

- $\pi_1(U_X \times_k \bar{k}) \stackrel{\text{def}}{=} \mathrm{Ker}(\pi_1(U_X) \twoheadrightarrow G_k)$

Then by using Belyi cuspidalization, we reconstruct

$$X_{\mathrm{NF}}^{\mathrm{cl}}, \quad \{\pi_1(U_X \setminus T) \twoheadrightarrow \pi_1(U_X)\}_{T \subseteq X_{\mathrm{NF}}^{\mathrm{cl}} :\text{ finite subset}}$$

By Nakamura, $\pi_1(U_X) \rightsquigarrow \{I_P \subseteq \pi_1(U_X)\}_{P \in (X \setminus U_X)^{\mathrm{cl}}}$

- $\pi_1(X) \stackrel{\text{def}}{=} \pi_1(U_X)/\overline{\langle I_P \mid P \in (X \setminus U_X)^{\mathrm{cl}} \rangle}$

Let $T \subseteq X_{\mathrm{NF}}^{\mathrm{cl}}$ be a finite subset

By Nakamura, $\pi_1(U_X \setminus T) \rightsquigarrow \{I_P \subseteq \pi_1(U_X \setminus T)\}_{P \in (X \setminus (U_X \setminus T))^{\mathrm{cl}}}$

$\rightsquigarrow \{D_P \stackrel{\text{def}}{=} N_{\pi_1(U_X \setminus T)}(I_P) \subseteq \pi_1(U_X \setminus T)\}_{P \in (X \setminus (U_X \setminus T))^{\mathrm{cl}}}$

$\rightsquigarrow \{G_{\kappa(P)} \stackrel{\text{def}}{=} D_P/I_P \subseteq \pi_1(X)\}_{P \in (X \setminus (U_X \setminus T))^{\mathrm{cl}}}$

11 degree map

- $\mathrm{Div}(X^{\mathrm{cl}}_{\mathrm{NF}}) \overset{\mathrm{def}}{=} \bigoplus_{P \in X^{\mathrm{cl}}_{\mathrm{NF}}} \mathbb{Z}$
- $\deg\colon \mathrm{Div}(X^{\mathrm{cl}}_{\mathrm{NF}}) \to \mathbb{Z}\colon \sum n_P \cdot P \mapsto \sum n_P \cdot [G_k : G_{\kappa(P)}]$
- $X^{\mathrm{cl}}_{\mathrm{NF}}(k) \overset{\mathrm{def}}{=} \{P \in X^{\mathrm{cl}}_{\mathrm{NF}} \mid \deg(P) = 1\}$

Note that $X^{\mathrm{cl}}_{\mathrm{NF}}(k) \overset{\mathrm{def}}{=} X^{\mathrm{cl}}_{\mathrm{NF}} \cap X(k)$

Then by using Belyi cuspidalization and deg, we reconstruct

$$\{\pi_1(U_X \setminus S) \twoheadrightarrow \pi_1(U_X) \twoheadrightarrow \pi_1(X) \twoheadleftarrow \pi_1(X \setminus S)\}_{S \subseteq X^{\mathrm{cl}}_{\mathrm{NF}}(k)\colon \text{finite subset}}$$

12 Principal divisors

- $\pi_1(X_{\bar{k}}) \stackrel{\text{def}}{=} \mathrm{Ker}(\pi_1(X) \twoheadrightarrow G_k)$
- $\pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \stackrel{\text{def}}{=} \pi_1(X_{\bar{k}})^{\mathrm{ab}}$
- $\pi_1(\mathrm{Pic}^1_X) \stackrel{\text{def}}{=} \pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \coprod_{\pi_1(X_{\bar{k}})} \pi_1(X)$
- $\pi_1(\mathrm{Pic}^2_X) \stackrel{\text{def}}{=} \pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \coprod_{\pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \times \pi_1(\mathrm{Pic}^1_{X_{\bar{k}}})} (\pi_1(\mathrm{Pic}^1_X) \times_{G_k} \pi_1(\mathrm{Pic}^1_X))$

where $\pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \times \pi_1(\mathrm{Pic}^1_{X_{\bar{k}}}) \to \pi_1(\mathrm{Pic}^1_{X_{\bar{k}}})$ denotes the multiplication
we define $\pi_1(\mathrm{Pic}^n_X)$ ($n \in \mathbb{Z}$) as well as

- $\pi_1(\mathrm{Pic}^n_{X_{\bar{k}}}) \stackrel{\text{def}}{=} \mathrm{Ker}(\pi_1(\mathrm{Pic}^n_X) \twoheadrightarrow G_k)$

We obtain

$$\{s_P \colon D_X \subseteq \pi_1(X \setminus S) \twoheadrightarrow \pi_1(X) \twoheadrightarrow \pi_1(\mathrm{Pic}^1_X)\}_{P \in (X \setminus (U_X \setminus S))^{\mathrm{cl}}}$$

Let $D \in \mathrm{Div}(X^{\mathrm{cl}}_{\mathrm{NF}}(k))$, if $\deg(D) = 0$, then $s_D \in \mathrm{H}^1(G_k, \pi_1(\mathrm{Pic}^0_{X_{\bar{k}}}))$

Lemma ([AbsTpIII], (1.6))
D: principal $\stackrel{\text{def}}{\iff} \exists f \in \mathrm{Fnct}(X)^\times$ s.t. $\mathrm{div}(f) = D$
$\iff \deg(D) = 0$, and $s_D = 0$ in $\mathrm{H}^1(G_k, \pi_1(\mathrm{Pic}^0_{X_{\bar{k}}}))$

We obtain $\mathrm{PDiv}(X^{\mathrm{cl}}_{\mathrm{NF}}(k)) \stackrel{\text{def}}{=} \{D \in \mathrm{Div}(X^{\mathrm{cl}}_{\mathrm{NF}}(k)) \mid D \text{ is principal}\}$

13 Synchronization of geometric cyclotomes

[AbsTpIII], (1.4)

- $\Lambda_X \stackrel{\text{def}}{=} \mathrm{Hom}(\mathrm{H}^2(\pi_1(X_{\bar{k}}), \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$

Note that $\Lambda_X \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$

- $\{\pi_1(U_X \setminus \{P\}) \twoheadrightarrow \pi_1(U_X) \twoheadrightarrow \pi_1(X) \twoheadleftarrow \pi_1(X \setminus \{P\})\}_{P \in X^{\mathrm{cl}}_{\mathrm{NF}}(k)}$ (cf. p.11 $S = \{P\}$)

Let $P \in X^{\mathrm{cl}}_{\mathrm{NF}}(k)$

$\pi_1(X \setminus \{P\}) \twoheadrightarrow G_k \rightsquigarrow I_P \subseteq \pi_1(X \setminus \{P\})$ (By Nakamura)

$\rightsquigarrow \pi_1((X \setminus \{P\})_{\bar{k}}) \stackrel{\text{def}}{=} \mathrm{Ker}(\pi_1(X \setminus \{P\}) \twoheadrightarrow G_k)$

$\rightsquigarrow 1 \to I_P \to \pi_1^{\mathrm{cc}}((X \setminus \{P\})_{\bar{k}}) \to \pi_1(X_{\bar{k}}) \to 1$

Note that $\pi_1((X \setminus \{P\})_{\bar{k}}) \to \pi_1^{\mathrm{cc}}((X \setminus \{P\})_{\bar{k}})$ is the maximal cuspidally central quotient

$\rightsquigarrow E_2^{i,j} = \mathrm{H}^i(\pi_1(X_{\bar{k}}), \mathrm{H}^j(I_P, I_P)) \Longrightarrow \mathrm{H}^{i+j}(\pi_1^{\mathrm{cc}}((X \setminus \{P\})_{\bar{k}}), I_P)$

$\rightsquigarrow \mathrm{Hom}(I_P, I_P) = \mathrm{H}^0(\pi_1(X_{\bar{k}}), \mathrm{H}^1(I_P, I_P)) \to_{d^{0,1}} \mathrm{H}^2(\pi_1(X_{\bar{k}}), \mathrm{H}^0(I_P, I_P)) = \mathrm{Hom}(\Lambda_X, I_P)$

$\rightsquigarrow d^{0,1}(\mathrm{id})\colon \Lambda_X \xrightarrow{\sim} I_P$

We obtain

$$\{d^{0,1}(\mathrm{id})\colon \Lambda_X \xrightarrow{\sim} I_P \subseteq \pi_1(X \setminus \{P\})\}_{P \in X^{\mathrm{cl}}_{\mathrm{NF}}(k)}$$

14 Kummer Theory-1

$$
\begin{array}{ccccc}
\mathrm{H}^1(G_k, \Lambda_X) & \hookrightarrow & \mathrm{H}^1(\pi_1(X \setminus S), \Lambda_X) & \longrightarrow & \bigoplus_{P \in S} \mathrm{H}^1(I_P, \Lambda_X) \\
\| \ \text{Kummer theory} & & \| & & \| \\
\widehat{k^\times} & \hookrightarrow & \widehat{\mathcal{O}^\times(X \setminus S)} & \longrightarrow & \bigoplus_{P \in S} \widehat{\mathbb{Z}} \\
\| & & \cup & & \cup \\
\widehat{k^\times} & \hookrightarrow & \widehat{k^\times} \cdot \mathcal{O}^\times_{\mathrm{NF}}(X \setminus S) & \longrightarrow & \mathrm{PDiv}(S) \\
\text{sub-}p\text{-adic} \ \cup & & \cup & & \| \\
k^\times & \hookrightarrow & \mathcal{O}^\times(X \setminus S) & \longrightarrow & \mathrm{PDiv}(S) \\
\cup & & \cup & & \| \\
k_0^\times & \hookrightarrow & \mathcal{O}^\times_{\mathrm{NF}}(X \setminus S) & \longrightarrow & \mathrm{PDiv}(S)
\end{array}
$$

We want to reconstruct $k_0^\times$, $\mathcal{O}^\times_{\mathrm{NF}}(X \setminus S)$

We use "container" $\mathrm{H}^1(\pi_1(X \setminus S), M)$

15 Kummer Theory-2

Let $S \subseteq X_{\mathrm{NF}}^{\mathrm{cl}}(k)$: a finite subset

By Hochschild-Serre spectral sequence

$$\mathrm{H}^1(G_k, \Lambda_X) \hookrightarrow \mathrm{H}^1(\pi_1(X \setminus S), \Lambda_X) \longrightarrow \bigoplus_{P \in S} \mathrm{H}^1(I_P, \Lambda_X)$$

- $\widehat{k^\times} \overset{\mathrm{def}}{=} \mathrm{H}^1(G_k, \Lambda_X)$ (By Kummer theory) (cf. [AbsTpIII], (1.6))

Note that $\mathrm{H}^1(I_P, \Lambda_X)$ is isom. to $\hat{\mathbb{Z}}$

- $\mathrm{PDiv}(S) \overset{\mathrm{def}}{=} (\bigoplus_{P \in S} \mathbb{Z}) \cap \mathrm{PDiv}(X_{\mathrm{NF}}^{\mathrm{cl}}(k))$

We obtain $\mathrm{PDiv}(S) \subseteq \bigoplus_{P \in S} \mathbb{Z} \hookrightarrow \bigoplus_{P \in S} \mathrm{H}^1(I_P, \Lambda_X)$

where $\mathbb{Z} \to \mathrm{H}^1(I_P, \Lambda_X) \mid 1 \mapsto d^{0,1}(\mathrm{id})^{-1}$

- $P_{X \setminus S} \overset{\mathrm{def}}{=} \mathrm{H}^1(\pi_1(X \setminus S), \Lambda_X) \times_{\bigoplus_{P \in S} \mathrm{H}^1(I_P, \Lambda_X)} \mathrm{PDiv}(S)$ (fiber-product)

We obtain

$$\widehat{k^\times} \hookrightarrow P_{X \setminus S} \longrightarrow \mathrm{PDiv}(S)$$

Note that $P_{X \setminus S} = \widehat{k^\times} \cdot \mathcal{O}_{\mathrm{NF}}^\times(X \setminus S)$

16 Kummer Theory-3

[AbsTpIII], (1.6)

- $\widehat{\kappa(P)^\times} \overset{\text{def}}{=} \mathrm{H}^1(D_P^{\text{cpt}}, \Lambda_X)$ (By Kummer theory)

  Evaluation section $D_P^{\text{cpt}} \to \pi_1(X \setminus S)$ induces

$$P_{X\setminus S} \subseteq \mathrm{H}^1(\pi_1(X \setminus S), \Lambda_X) \to \mathrm{H}^1(D_P^{\text{cpt}}, \Lambda_X) = \widehat{\kappa(P)^\times}$$

$$f \mapsto f(P)$$

- $\mathcal{O}_{\text{NF}}^\times(X \setminus S) \overset{\text{def}}{=} \{f \in P_{X\setminus S} \mid \exists P \in X_{\text{NF}}^{\text{cl}} \setminus S, \exists n \in \mathbb{Z}_{>0} : f(P)^n = 1 \in \widehat{\kappa(P)^\times}\}$

17 Kummer Theory-4

Let $H \subseteq G_k$: open

$\{\pi_1(U_X \times_k k') \twoheadrightarrow G_{k'}\}_{k'/k:\text{ finite}} \overset{\text{def}}{=} \{\mathrm{Aut}(\pi_1(U_X \times_k \bar{k})) \times_{\mathrm{Out}(\pi_1(U_X \times_k \bar{k}))} H \twoheadrightarrow H\}_H$

Let $\pi_1(U_X \times_k k') \twoheadrightarrow G_{k'}$

By the same argument, we reconstruct

- $(X \times_k k')^{\mathrm{cl}}_{\mathrm{NF}}$
- $\mathcal{O}^{\times}_{\mathrm{NF}}((X \setminus S)_{k'})$

Note that $(X \times_k k')^{\mathrm{cl}}_{\mathrm{NF}} \overset{\text{def}}{=} \mathrm{Im}((X_0 \times_{k_0} k'_0)^{\mathrm{cl}} \hookrightarrow (X \times_k k')^{\mathrm{cl}})$

where k'_0 denotes the algebraic closure of k_0 in k'

18 Kummer Theory-5

- $(X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \varinjlim_{k'} (X \times_k k')^{\mathrm{cl}}_{\mathrm{NF}}$
- $\mathcal{O}^\times((X_0 \setminus S_0)_{\bar{k}_0}) \stackrel{\mathrm{def}}{=} \varinjlim_{k'} \mathcal{O}^\times_{\mathrm{NF}}((X \setminus S)_{k'})$

Note that

$$\begin{array}{ccccc}
X_0^{\mathrm{cl}} & \xrightarrow{\sim} & X^{\mathrm{cl}}_{\mathrm{NF}} & \hookrightarrow & X^{\mathrm{cl}} \\
 & & \cup & & \\
\cup & & X^{\mathrm{cl}}_{\mathrm{NF}}(k) & & \\
 & & \cup & & \\
S_0 \stackrel{\mathrm{def}}{=} (X_0^{\mathrm{cl}} \to X^{\mathrm{cl}}_{\mathrm{NF}})^{-1}(S) & \xrightarrow{\sim} & S & &
\end{array}$$

- $\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times \stackrel{\mathrm{def}}{=} \varinjlim_{S} \mathcal{O}^\times((X_0 \setminus S_0)_{\bar{k}_0})$: a multiplicative group

We want to reconstruct the field structure of $\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times$

19 Order maps, divisor maps, and evaluation

$$\mathrm{H}^1(G_{k'}, \Lambda_X) \hookrightarrow \mathrm{H}^1(\pi_1((X \setminus \{P\})_{k'}), \Lambda_X) \longrightarrow \mathrm{H}^1(I_P, \Lambda_X)$$

(cf. p.15)

$$\mathcal{O}^\times_{\mathrm{NF}}((X \setminus \{P\})_{k'}) \subseteq \mathrm{H}^1(\pi_1((X \setminus S)_{k'}), \Lambda_X) \to \mathrm{H}^1(I_P, \Lambda_X)$$

induces

$$\mathrm{ord}_P \colon \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times \to \mathbb{Z} \ \ (\hookrightarrow \mathrm{H}^1(I_P, \Lambda_X))$$

- $\bar{k}_0^\times \overset{\mathrm{def}}{=} \bigcap_P \mathrm{Ker}(\mathrm{ord}_P) \subseteq \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times$
- $\mathrm{div} \colon \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times \to \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}) \colon f \mapsto \sum_P (\mathrm{ord}_P(f)) \cdot P$
- evaluation (cf. p.16)

$$P_{X \setminus S} \subseteq \mathrm{H}^1(\pi_1(X \setminus S), \Lambda_X) \to \mathrm{H}^1(D_P^{\mathrm{cpt}}, \Lambda_X) \overset{\sim}{\to} \widehat{\kappa(P)^\times}$$

induces

$$\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^\times \to \bar{k}_0^\times \sqcup \{0\} \sqcup \{\infty\}$$

$$f \mapsto f(P)$$

20 Already reconstructed

START: $\pi_1(U_X) \twoheadrightarrow G_k$

We already reconstructed

$$((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}},\ \bar{k}_0^{\times},\ \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times},$$

$$\mathrm{ord}_P \colon \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times} \to \mathbb{Z},$$

$$\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times} \to \bar{k}_0^{\times} \sqcup \{0\} \sqcup \{\infty\} \colon f \mapsto f(P))$$

Next, by Uchida, we reconstruct the field structure of $\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times}$

21 Uchida-1

We know multiplicative structure of $\bar{k}_0 \overset{\text{def}}{=} \bar{k}_0^\times \sqcup \{0\}$

- $1 \in \bar{k}_0^\times$: the unit element
- $-1 \overset{\text{def}}{=} a \in \bar{k}_0^\times \setminus \{1\}$ is a unique element s.t. $a^2 = 1$

Note that $\mathrm{ch}(\bar{k}_0) = 0$

Let $a, b \in \bar{k}_0^\times$ s.t. $a \neq -1 \cdot b$

Then we want to reconstruct $a + b \in \bar{k}_0^\times$

22 Uchida-2

Note that we know additive structure of $\mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$

Let $D \in \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$

- $\mathrm{Div}^+ \stackrel{\mathrm{def}}{=} \mathrm{Div}^+((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}) \stackrel{\mathrm{def}}{=} \{\sum_x n_x \cdot x \mid n_x \geq 0\} \subseteq \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$
- $\mathrm{H}^0(D) \stackrel{\mathrm{def}}{=} \{f \in \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0) \mid \mathrm{div}(f) + D \in \mathrm{Div}^+\} \cup \{0\}$
- $\mathrm{h}^0(D) \stackrel{\mathrm{def}}{=} \min\{n \mid \exists E \in \mathrm{Div}^+,\ \deg(E) = n,\ \mathrm{H}^0(D - E) = 0\}$

23 Uchida-3

Proposition ([AbsTpIII], (1.2))

$\exists D \in \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$, $\exists P_1, P_2, P_3 \in (X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}$ distinct points s.t. the following hold:

(i) $\mathrm{h}^0(D) = 2$

(ii) $P_1, P_2, P_3 \notin \mathrm{Supp}(D) \overset{\mathrm{def}}{=} \{x \mid D = \sum n_x \cdot x, n_x \neq 0\}$

(iii) $\mathrm{h}^0(D - P_i - P_j) = 0 \quad (\forall i \neq j \in \{1, 2, 3\})$

Let $D \in \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$, $P_1, P_2, P_3 \in (X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}$ distinct points which satisfy (i), (ii), (iii)

24 Uchida-4

Proposition [AbsTpIII], (1.2)

$a, b \in \bar{k}_0^\times$ s.t. $a \neq -1 \cdot b$

$\exists ! f \in \mathrm{H}^0(D),\ f(P_1) = 0,\ f(P_2) \neq 0,\ f(P_3) = a$

$\exists ! g \in \mathrm{H}^0(D),\ f(P_1) \neq 0,\ f(P_2) = 0,\ f(P_3) = b$

$\exists ! h \in \mathrm{H}^0(D),\ h(P_1) = g(P_1),\ h(P_2) = f(P_2)$

In particular, $h = f + g$ and $a + b = h(P_3)$

Proof. we consider f

Note that $\mathrm{H}^0(D - P_1 - P_2) = 0 \subsetneq \mathrm{H}^0(D - P_1) \subsetneq \mathrm{H}^0(D)$

existence: $f_0 \in \mathrm{H}^0(D - P_1) \setminus \{0\}$

by (ii), $f_0(P_1) = 0$

by (ii), (iii), $f_0(P_2), f_0(P_3) \in \bar{k}_0^\times$

$f \stackrel{\text{def}}{=} \frac{a}{f(P_3)} f_0 \in \mathrm{H}^0(D - P_1) \subsetneq \mathrm{H}^0(D)$

uniqueness and h: by (iii)

25 Uchida-5

Let $a, b \in \bar{k}_0^\times$ s.t. $a \neq -1 \cdot b$

We want to reconstruct $a + b \in \bar{k}_0^\times$

Let $D \in \mathrm{Div}((X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}})$, $P_1, P_2, P_3 \in (X_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}$, and $f, g, h \in \mathrm{H}^0(D)$

- $a + b \stackrel{\mathrm{def}}{=} h(P_3)$

Thus, we reconstruct the field structure of $\bar{k}_0$

26 Uchida-6

$\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0) \stackrel{\mathrm{def}}{=} \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times} \sqcup \{0\}$

Let $f, g \in \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)$

We want to reconstruct $f + g \in \mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)$

Two different proofs:

- Identity theorem (Note that $\mathrm{Supp}(f) \cup \mathrm{Supp}(g) \cup \mathrm{Supp}(f + g)$ is finite)
- $\mathcal{O}^{\times}((X_0 \setminus S_0)_{\bar{k}_0}) \hookrightarrow \prod \bar{k}_0$

$\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)^{\times} \stackrel{\mathrm{def}}{=} \varinjlim_S \mathcal{O}^{\times}((X_0 \setminus S_0)_{\bar{k}_0})$

Thus, we reconstruct the field structure of $\mathrm{Fnct}(X_0 \times_{k_0} \bar{k}_0)$

27 Supplement-1

K: an algebraic closure

X/K: a proper smooth curve

Then

$$(X^{\mathrm{cl}},\ K^\times,\ \mathrm{Fnct}(X)^\times,$$

$$\mathrm{ord}_P \colon \mathrm{Fnct}(X)^\times \to \mathbb{Z},$$

$$\mathrm{Fnct}(X)^\times \to K^\times \sqcup \{0\} \sqcup \{\infty\} \colon f \mapsto f(P))$$

$\rightsquigarrow$ the field structure of $\mathrm{Fnct}(X)^\times$

<u>Remark</u> We can reconstruct if K is infinite (Higashiyama)

28 Supplement-2

There exist 2 type of mono reconstruction of additive structures, I think

- Using Uchida' Lemma

We need base fields K and function fields $\mathrm{Fnct}(X)$

- Using $(0,4)$

(i) Higashiyama: Let U_X: $(0,3)$ curve

Then we consider the second configuration space $(U_X)_2 \subseteq (0,3) \times (0,3)$

Since $(U_X)_2$ is nearly equal to $(0,4) \times (0,3)$

$(0,4) \rightsquigarrow$ the additive structure of K

(ii) Hoshi: Let U_X: $(0,3)$ curve

By Belyi cuspidalization, we obtain many $(0,4)$ curves

$(0,4) \rightsquigarrow$ the additive structure of K

References

[AbsTpI] S. Mochizuki, Topics in Absolute Anabelian Geometry I: Generalities, *J. Math. Sci. Univ. Tokyo.* **19** (2012), 139-242.

[AbsTpII] S. Mochizuki, Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms, *J. Math. Sci. Univ. Tokyo.* **20** (2013), 171-269.

[AbsTpIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo.* **22** (2015), 939-1156.